An Approach to the Theory of Multidimensional Elliptic Curves

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Abstract

In this paper, we will be presenting an approach to generalizing the theory of elliptic curves. This paper, to summarize broadly, will be the documentation of the results of a research project centered around a certain generalization of elliptic curves, namely a "multidimensional elliptic curve".

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1 Introductory Theory

An elliptic curve, E, in the literature, is a plane curve for a field, K, typically defined as: $\{(x, y)|y^2 = x^3 + ax + b\}$*, for given coefficients* $a, b \in K$*.*

Def. 1.1. *An "index (n, k)-Borcherds surface" is an elliptic curve defined by the series in* L^k *for a field, L:*

$$x_{n+1}^2 = \sum_{i=1}^{n} (x_i^k + a_{1,i}x_i^{k-2} + ... + a_{k-1,i})$$

When the "index" is (1,3), we see it reduce to the typical elliptic curve,

$$x_2^2 = x_1^3 + a_1x_1 + a_2$$

. We can formally realize this approach using parameter spaces. When we have specific values of n and k, the parameter space which we will notate $Bor(n\colon k)$ *is considered, which we construct as the product:*

$$Bor(n : k) = \prod_{i=1}^{n} V_{k-1}(S)$$

where S is the set our elliptic curve's coefficients rest in. The "parameter space of all Borcherds surfaces" can be realized as the product:

$$\bigoplus_{(n,k)\in\mathbb{Z}^+\times\mathbb{Z}^+_{\geq 2}} Bor(n:k)$$

. We consider this space to be "2-dimensional", or, that the projections have 2 indices e.g. like a matrix. This is for mathematical convenience.

2 Results of Research

There are many results/definitions that come from the theory of 2-dimensional elliptic curves which we can generalize to this theory:

Def. 2.1. *A (k, 3)-Borcherds surface, $\mathcal{B}$ is said to have good reduction at a prime, p, if $\Delta\mathcal{E}_i(\mathcal{B} \mod p) \neq 0\ \forall i \in \{1, ...k-1\}$.*

Def. 2.2. *Similarly, a (k, 3)-Borcherds surface, $\mathcal{B}$ is said to have bad reduction at a prime, p, if $\exists i$ s.t.$\Delta\mathcal{E}_i(\mathcal{B} \mod p) = 0$.*

Def. 2.3. *The discriminant of a (k, 3)-Borcherds surface is the product of the discriminants of its individual "projections". Or,*
$\Delta_{\mathcal{B}} = \prod_{i=1}^{k} \Delta(\mathcal{E}_i(\mathcal{B}))$.
This preserves a key property in the study of elliptic curves behind discriminants, that of it being zero if there are repeated roots in $x^3 + ax + b$ for an elliptic curve $\mathcal{E} : y^2 = x^3 + ax + b$.

Def. 2.4. *The global conductor of a Borcherds surface, $\mathcal{B}$ is the product of the global conductors of its "projections". This preserves* **Def. 2.1-2.3.**.
Expressed in notation:
$Cond(\mathcal{B}) = \prod_{i=1}^{k} Cond(\mathcal{E}_i(\mathcal{B}))$
The general case for any Borcherds surface, $\mathcal{B}$ is harder to define, but we can define such a generalization of the global conductor "naively", which we can realize as the product of the conductors of the individual "projections" of $\mathcal{B}$.
We may also define such a generalization using the notion of a "conductor" for a general abelian variety. This may be more suitable for those with a background in algebraic geometry.
These results are much different from those contained in [SS16]. Additionally, there may exist theories of multi-dimensional elliptic curves which may have alternate but consistent definitions of the discriminant and others.

Thm. 2.1. *(Hasse) For a finite field, $\mathbb{F}_p$, and an elliptic curve, $\mathcal{E}$, $|\#\mathcal{E}(\mathbb{F}_p) - (p+1)| \leq 2\sqrt{p}$.*
We can generalize this theorem to a Borcherds surface, $\mathcal{B}$ of index (k, 3), since we know it is the sum of k elliptic curves, all of which satisfy this bound.

Thm. 2.2. *(Hasse for Borcherds surfaces) For a finite field, $\mathbb{F}_p$, and a (k, 3)-Borcherds surface, $\mathcal{E}$, $|\#\mathcal{B}(\mathbb{F}_p) - (p+1)| = \sum_{i=1}^{k}(|\#\mathcal{E}_i^{\mathcal{B}}(\mathbb{F}_p)|) \leq 2k\sqrt{p}$.*

Proof. `This is trivial to prove.` *There is a specific way we can define "addition" over (k, 3)-Borcherds surfaces. We will start with the original definition of addition on elliptic curves for reference.*

Def. 2.5. *Given an elliptic curve, $\mathcal{E}$ in the projective space $\mathbb{P}^2$, we define addition for a pair of points $\{a, b\} \subset \mathcal{B}$ as $a + b = -(\mathcal{E} \cap (\mathfrak{l}_{a,b} - \{a, b\}))$. Here, $\mathfrak{l}_{a,b}$*

is the secant line on the elliptic curve.
The above definition is immediately generalizable to higher dimensions, because of the LHS having an exponent of 2, therefore it has a symmetry over the coordinate x_{k+1} which is analogous to that of addition for (1, 3)-Borcherds surfaces, or, classical elliptic curves.
Def. 2.6. *Given a Borcherds surface, $\mathcal{B}$ in the projective space $\mathbb{P}^{k+1}$, defined over homogeneous coordinates. For $\{a, b\} \subset \mathcal{B}$, we define the "addition" $a + b = -(\mathcal{B} \cap (\mathfrak{l}_{a,b} - \{a, b\}))$. We follow a similar rule to the case where $k = 1$, where we define $-a = (\pi_1(a), ..., \pi_k(a), -\pi_{k+1}(a))$.*
A generalization for the "algebraic" interpretation of addition is not immediately obvious, as singular number slopes are only possible in 2 dimensions. Such an interpretation will be touched on in a later paper.
When $k = \infty$, or, when we consider the cases as $k \to \infty$, we immediately recognize that **Thm. 2.2** *breaks, and so does* **Thm. 2.3.** *as an immediate consequence. We can also no longer conveniently define objects such as the conductor or discriminant as we did earlier in this paper. This means the theory of (∞, n)-Borcherds surfaces is incredibly restrictive and may not be of much arithmetic interest. The theory of (∞, ∞)-Borcherds surfaces is, however, completely nonexistent. If we substitute ∞ and ∞ into* **Def. 1.1.**, *we contain a nonsensical expression, which the way to even begin to salvage it is unclear.*
Aside from ordinary elliptic curves, we can generalize other types of elliptic curves to become multidimensional, to create different "types" of Borcherds surfaces. These will be explored in later papers on this subject.

3 Works Referenced or Utilized/Further Reading

[SS16], Sonnino, Sonnino - Elliptic-Curves Cryptography on High-Dimensional Surfaces